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Applications of mathematics in selected control and decision processes

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1. Introduction. Rapid development of computer science in recent years allowed more detailed analysis and synthesis of control systems for complex processes and supports decision making in different practical areas. Discoveries in mathematical nature of the universe stimulate representatives of technical sciences to actions leading into planned affecting of real objects. Practical verification of engineers ideas in many cases is effective and leads to meaningful results.

In control of dynamical systems the following algorithm of operations one has proved to be effective (see for example [27, 28, 32, 33])

1. Create a mathematical model, usually in a form of an appropriate differential equation.
2. Perform the linearisation of the model.
3. Design a control system, for example through appropriate feedback - usually through formulation of some kind of LQ problem.
4. Verify the design with a real life system.

It should be noted however, that control problems are not limited to applications of this algorithm. For example the parameters of the constructed model have to be obtained through the identification. We require that the designed control systems have certain properties. Most notable is the aspect of asymptotic (exponential) stability of the system. Different notions of stability are used, but most popular is the Lyapunov stability, also important is the practical stability. Along with stability also the aspect of area (basin) of attraction is discussed usually in context of LaSalle principle [22] (also known as Krasovskii-LaSalle principle). It is also desired, that the designed control system would possess such typical properties known from control theory as controllability and observability (stabilisability and detectability).

In many cases not all needed measurements are available. In such case if system is observable one can construct a state observer - a dynamical system which estimates the unmeasured state variables. In other cases practical realisation of control systems requires application of computers or embedded circuits in real time regimes. In such cases an important aspect is the operation of appropriate A/D (analog/digital) and D/A (digital/analog) converters – their synchronisation, their sampling frequency. Also the spatial placement of sensors (distance between them) should also be considered.

Determination of control signal also is an interesting aspect. In most cases it is desired that the control should have a form of feedback. Often feedback can be designed using appropriate Lyapunov and Riccati equations (see for example [1] or [26, 27, 21]) usually because of the connection to the LQ problem (see [17]) and optimal filtration problem (see for example [9]). In other cases however different methods can be used. Stabilising feedback can be constructed through a construction of appropriate Lyapunov function or by influencing the location of system's eigenvalues. Feedback can also be designed by solving appropriate game theory problem, for example for LQ games. Moreover not all control problems have a structure of feedback – in some cases control can be given as a function of time (so called open loop control) which will be a solution to certain dynamical optimisation problems (for example time optimal control).

In this paper we present a series of examples showing different applications of control theory and game theory to different systems. Substantial part of them are the stabilisation problems but there are also state estimation, identification, optimal control, shape optimisation and decision support through game theory.

2. DC motor control. Direct current (DC) machines are very popular in practical applications. In figure 1 a diagram of a typical separately excited DC motor is presented along with physical variables and constants which are used in the mathematical model.

The separately excited DC motor can be described by using the following three nonlinear differential equations [5, 23]:

$$
\begin{aligned}
L_{\mathrm{t}}\frac{\mathrm{d}i_{\mathrm{t}}(t)}{\mathrm{d}t} &= u_{\mathrm{t}}(t) - R_{\mathrm{t}}i_{\mathrm{t}}(t) - c_{\mathrm{E}}\phi_{\mathrm{w}}(t)\omega(t)\\
\frac{\mathrm{d}\phi_{\mathrm{w}}(t)}{\mathrm{d}t} &= -R_{\mathrm{w}}f^{-1}(\phi_{\mathrm{w}}(t)) + u_{\mathrm{w}}(t)\\
M_{\mathrm{J}}\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} &= c_{\mathrm{M}}\phi_{\mathrm{w}}(t)i_{\mathrm{t}}(t) - B_{\mathrm{v}}\omega(t) - M_{\mathrm{Z}}
\end{aligned}
\tag{1}
$$

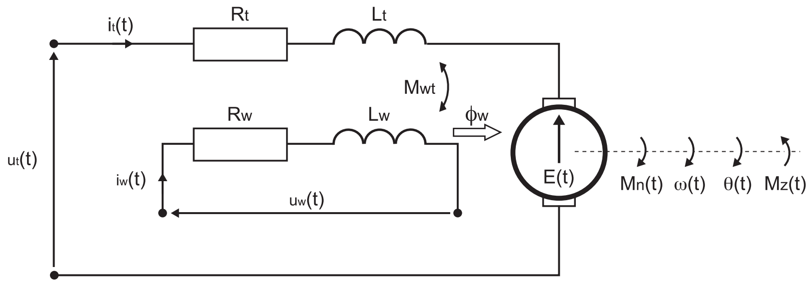

Figure 1: Diagram of a separately excited DC machine.

2.1. *Methods of linear control.* Because system (1) is nonlinear certain methods of analysis cannot be used. However, equations of (1) can be simplified by using the following assumptions:

- magnetic flux of a stator circuit is constant $\phi_{\mathrm{w}}(t) = const$ and magnetic fluxes of armature and stator circuits are not coupled,
- all physical parameters (e.g. resistance or inductance) are not varying with time and are not dependent on temperature or position,
- only viscosity friction is considered and is modelled as linear function of angle velocity.

Under these assumptions we can describe DC machine using the following system of two linear differential equations [37]

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{Z}\mathbf{z}(t) \tag{2}$$

where appropriate matrixes and vectors are:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} -\frac{R_{\mathrm{t}}}{L_{\mathrm{t}}} & -\frac{K_{\mathrm{m}}\phi_{\mathrm{w}}}{L_{\mathrm{t}}} \\ \frac{K_{\mathrm{m}}\phi_{\mathrm{w}}}{M_{\mathrm{J}}} & -\frac{B_{\mathrm{v}}}{M_{\mathrm{J}}} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \frac{1}{L_{\mathrm{t}}} \\ 0 \end{bmatrix}, \ \mathbf{Z} = \begin{bmatrix} 0 \\ -\frac{1}{M_{\mathrm{J}}} \end{bmatrix} \tag{3}$$

and state space vector $\mathbf{x}(t)$, contains two elements $x_1(t) = i_{\mathrm{t}}(t)$, $x_2(t) = \omega(t)$. Load torque M_{Z} in this model is considered as a disturbance.

Next step, after obtaining the mathematical model of DC motor, is the identification of model parameters. Some parameters can be measured directly, e.g. resistance. Other parameters must be identified from more than one measurement, e.g. inductance. Also some parameters can only be approximated, for example inertia moment of shaft or the friction coefficient. Figures 2 and 3 present measured (gray line) and simulated (black line) current and angular velocity of real DC machine. As it can be seen outputs from mathematical model, current and angular velocity, are very similar to the corresponding outputs real DC machine [13].

Let us consider stabilisation of the angular velocity $\omega(t)$ of a DC machine on a constant level ω_Z while the load torque M_{Z} is changed. If value of the load torque M_{Z} is changed then the angular velocity ω_Z of DC machine is also changed.

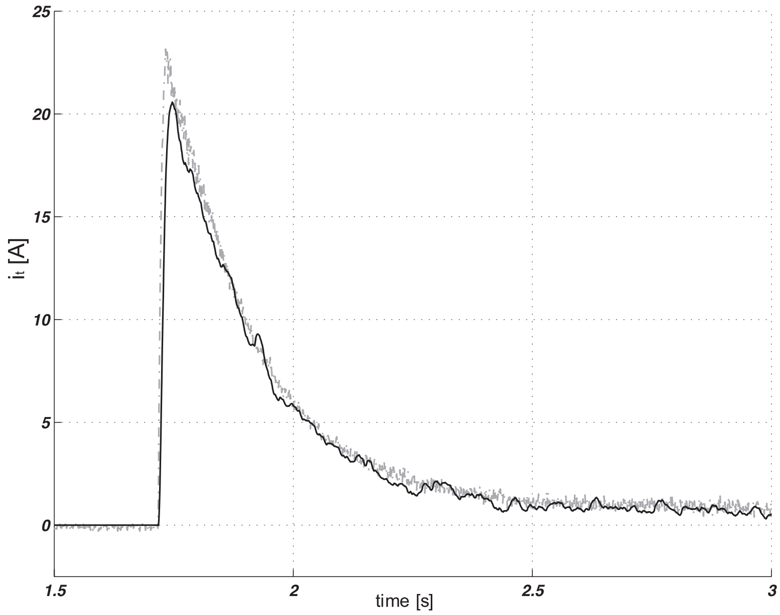

Figure 2: Comprarison real (gray line) and simulated current $i_{\mathrm{t}}(t)$ (black line) of DC machine.

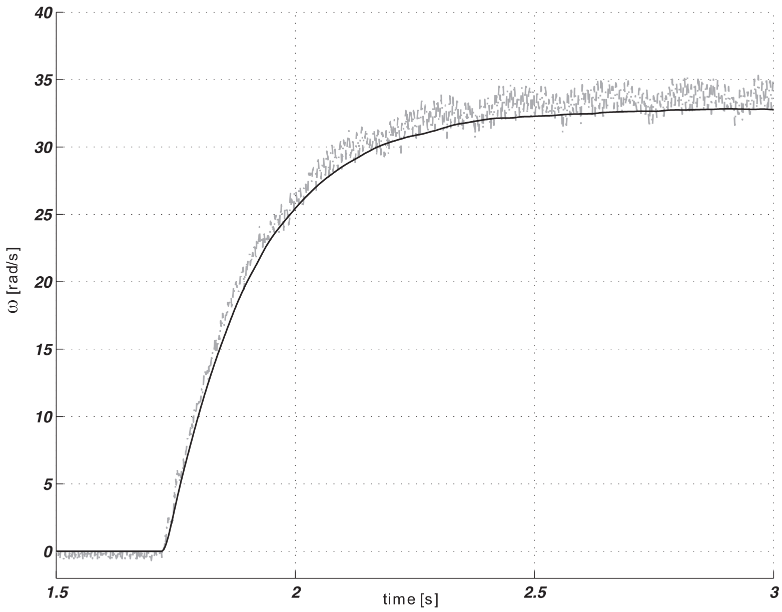

Figure 3: Comprarison real (gray line) and simulated angular velocity $\omega(t)$ (black line) of DC machine.

Let us use a proportional controller given by the following equation:

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t) \tag{4}$$

Matrix $\mathbf{K}$ is defined as

$$\mathbf{K} = -\mathbf{R}^{-1}\mathbf{B}^{\mathcal{T}}\mathbf{D} \tag{5}$$

where $\mathbf{D} = \mathbf{D}^{\mathcal{T}} = \mathbf{D} \geq 0$ is the unique, symetric, nonnegative solution of algebraic Riccati equation:

$$\mathbf{A}^{\mathcal{T}}\mathbf{D} + \mathbf{D}\mathbf{A} - \mathbf{D}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathcal{T}}\mathbf{D} + \mathbf{W} = 0 \tag{6}$$

where matrix $\mathbf{W}^{\mathcal{T}} = \mathbf{W} \geq 0$, matrix $\mathbf{R}^{\mathcal{T}} = \mathbf{R} > 0$. Solution of the algebraic Riccati equation, matrix $\mathbf{D}$, exists if the pair of matrices $(\mathbf{A}, \mathbf{B})$ is stabilisable and the pair $(\mathbf{W}, \mathbf{A})$ is detectable. The controller (4) is called the LQR controller and it is an optimal controller in the sense of quality index [17] [26]:

$$J(\mathbf{u}, \mathbf{x}^0) = \int_0^\infty (\mathbf{x}^{\mathcal{T}}(t)\mathbf{W}\mathbf{x}(t) + \mathbf{u}^{\mathcal{T}}(t)\mathbf{R}\mathbf{u}(t))\mathrm{d}t \tag{7}$$

where $\mathbf{x}_0$ is start point of the system (2). The pair of matrices $(\mathbf{A}, \mathbf{B})$ is stabilisable iff $\mathrm{rank}[s_i\mathbf{I} - \mathbf{A}\,\mathbf{B}] = n$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$ and s_i are eigenvalues of $\mathbf{A}$ with non negative real parts. The pair of matrices $(\mathbf{W}, \mathbf{A})$ is detectable iff the pair $(\mathbf{A}^{\mathcal{T}}, \mathbf{W}^{\mathcal{T}})$ is stabilisable.

The structure of a simple control system is shown in figure 4.

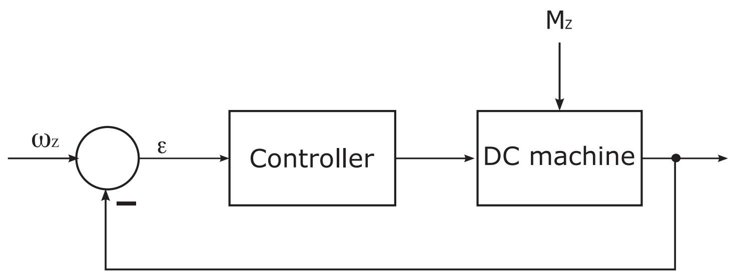

Figure 4: Diagram of control system stabilization angular velocity of DC machine.

By an appropriate choice of matrix $\mathbf{W}$ values we can decide which coordinate the state space vector $\mathbf{x}(t)$ is *better stabilized*. By an appropriate choice of matrix $\mathbf{R}$ values we can limit the maximum value $\mathbf{u}(t)$ what is important in practical applications. For practical experiment let us chose $\mathbf{W}$ and $\mathbf{R}$:

$$\mathbf{W} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$$

$$\mathbf{R} = [\,1\,]$$

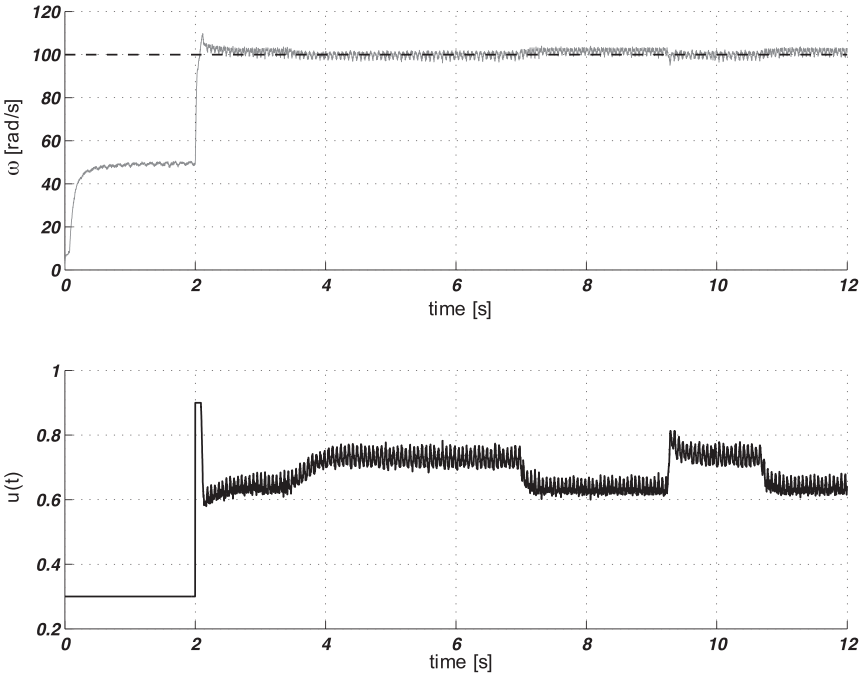

Figure 5: Angular velocity $\omega(t)$ of DC machine stabilised by LQR controller (upper plot) and control signal $\mathbf{u}(t)$ (bottom plot).

The upper plot in figure 5 shows a result of stabilization of the angular velocity of DC machine. Despite of changes to the load torque M_{Z} during experiment angular velocity is stabilised on the desired level $\omega_{\mathrm{Z}} = 100$ rad/s. The bottom plot in figure 5 shows how control signal $\mathbf{u}(t)$ was changing during the experiment (see [12] pp. 86–88).

2.2. *Methods of nonlinear control.* In this section we will consider separately excited DC motor in which the magnetic flux of the stator circuit is varying and is controlled. We will however introduce a different assumption. We will assume that for considered stator currents the magnetisation curve of the stator is linear. More specifically we will set

$$\phi_{\mathrm{w}}(t) = f(i_{\mathrm{w}}(t)) = L_{\mathrm{w}} i_{\mathrm{w}}(t) \tag{8}$$

Under this assumption system (1) becomes

$$\begin{aligned} \frac{\mathrm{d}i_{\mathrm{t}}(t)}{\mathrm{d}t} &= \frac{u_{\mathrm{t}}(t)}{L_{\mathrm{t}}} - \frac{R_{\mathrm{t}}}{L_{\mathrm{t}}} i_{\mathrm{t}}(t) - \frac{c_{\mathrm{E}} L_{\mathrm{w}}}{L_{\mathrm{t}}} i_{\mathrm{w}}(t)\omega(t) \\ \frac{\mathrm{d}i_{\mathrm{w}}(t)}{\mathrm{d}t} &= -\frac{R_{\mathrm{w}}}{L_{\mathrm{w}}} i_{\mathrm{w}}(t) + \frac{u_{\mathrm{w}}(t)}{L_{\mathrm{w}}} \end{aligned} \tag{9}$$

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{L_{\mathrm{w}}c_{\mathrm{M}}}{M_{\mathrm{J}}} i_{\mathrm{w}}(t) i_{\mathrm{t}}(t) - \frac{B_{\mathrm{v}}}{M_{\mathrm{J}}}\omega(t) - \frac{M_{\mathrm{Z}}}{M_{\mathrm{J}}}$$

Changing notation (and dropping the time argument) we can reformulate system (9) into

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 - a_2 x_2 x_3 + v_1 \\ \dot{x}_2 &= -b_1 x_2 + v_2 \\ \dot{x}_3 &= c_1 x_1 x_2 - c_2 x_3 - \tau \end{aligned} \tag{10}$$

where $x_1 = i_{\mathrm{t}}$, $x_2 = i_2$, $x_3 = \omega$, $v_1 = u_{\mathrm{t}}/L_{\mathrm{t}}$, $v_2 = u_{\mathrm{w}}/L_{\mathrm{w}}$ and $\tau = M_{\mathrm{Z}}/M_{\mathrm{J}}$. Rest of notation changes is self explanatory. This reformulated model will be used for describing applications of nonlinear control.

2.2.1. Nonlinear velocity observer. One of the many technical problems that can be answered by the application of mathematics is the problem of estimating the unmeasured variables. In particular this problem is especially important in the aspect of velocity measurement. In practical applications velocity can be obtained by three ways:

1. Direct measurement by specialized devices (tachogenerators).
2. Differentiation of position measurements (obtained via encoders or resolvers).
3. Integration of acceleration measurements (obtained from the accelerometer).

All these approaches require addition of specialized equipment, cost of which is substantial. Possibility of obtaining the velocity signal from other, already measured variables is then very beneficial. For the linear systems, there is a widely known theory of Luenberger observer, which allows state estimation from the output measurements. In nonlinear systems special techniques have to be applied.

Let us assume that measurements of both currents in the DC motor (10) are available from the measurements. Let us introduce the following change of coordinates

$$s_1 = \frac{x_1}{x_2},\ s_2 = x_2,\ s_3 = x_3 \tag{11}$$

Under the change of coordinates (11) system (10) becomes

$$\begin{aligned} \dot{s}_1 &= (b_1 - a_1)s_1 - a_2 s_3 + \frac{1}{s_2}v_1 - \frac{s_1}{s_2}v_2 \\ \dot{s}_2 &= -b_1 s_2 + v_2 \\ \dot{s}_3 &= c_1 s_1 s_2^2 - c_2 s_3 - \tau \end{aligned} \tag{12}$$

or in the vector matrix notation

$$\dot{\mathbf{s}} = \mathbf{A}\mathbf{s} + \mathbf{f}(s_1, s_2) + \mathbf{B}(s_1, s_2)\mathbf{v} + \mathbf{Z}\tau \tag{13}$$

where

$$\mathbf{A} = \begin{bmatrix} b_1 - a_1 & 0 & -a_2 \\ 0 & -b_1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix} \quad \mathbf{f}(s_1, s_2) = \begin{bmatrix} 0 \\ 0 \\ c_1 s_1 s_2^2 \end{bmatrix} \tag{14}$$

$$\mathbf{B}(s_1, s_2) = \begin{bmatrix} \frac{1}{s_2} & -\frac{s_1}{s_2} \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{15}$$

It should be noted, that because x_1 and x_2 were measurable also s_1 is measurable. We propose a following state observer

$$\dot{\hat{\mathbf{s}}} = \mathbf{A}\hat{\mathbf{s}} + \mathbf{f}(s_1, s_2) + \mathbf{G}\mathbf{C}(\mathbf{s} - \hat{\mathbf{s}}) + \mathbf{B}(s_1, s_2)\mathbf{v} + \mathbf{Z}\tau \tag{16}$$

where

$$\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{17}$$

What should be noted, is that if we introduce the error of estimation $\mathbf{e} = \mathbf{s} - \hat{\mathbf{s}}$ it evolves according to the following differential equation

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{G}\mathbf{C})\mathbf{e} \tag{18}$$

which is linear, and because pair $(\mathbf{C},\ \mathbf{A})$ is observable, by the choice of appropriate matrix G, eigenvalues of matrix $(\mathbf{A} - \mathbf{G}\mathbf{C})$ can be set as desired, in particular allowing exponential stability of error dynamics. The pair of matrices $(\mathbf{C},\ \mathbf{A})$ where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$ is observable iff $\operatorname{rank}[\lambda_i \mathbf{I} - \mathbf{A}^T \mathbf{C}^T] = n$ where λ_i are eigenvalues of $\mathbf{A}$ where $i = 1,\ 2,\ \ldots,\ n$ (equivalent definitions see [26]). This approach of using nonlinear transformations to design observers with linear error dynamics for electrical machines can be found in [10]. Application to different type of DC motor (series motor) see [2], optimisation of observer parameters see [4].

2.2.2. Time optimal control. Time optimal problem can be formulated as follows: "Find the control signal $\mathbf{v}^*$, for which system trajectories starting in point $\mathbf{x}_0$ will get to point $\mathbf{x}_k$ in minimal time". For linear systems this problem can be, at least on principle, solved directly. In nonlinear systems one usually solves a series of optimisation problems of finding $\mathbf{v}^*$ for which trajectories starting in the point $\mathbf{x}_0$ will get to the point $\mathbf{x}_k$ in given time T and then use these solutions to find the minimal T^*. Here we will present only the basics of search for such solution. We consider system (10), with controls subjected to constraints $v_1 \in [v_{1min}, v_{1max}]$ and $v_2 \in [v_{2min}, v_{2max}]$. We consider the performance index

$$Q(v) = q(x(T)) = (\mathbf{x}(T) - \mathbf{x}_k)^{\mathcal{T}}(\mathbf{x}(T) - \mathbf{x}_k) \tag{19}$$

To find the optimal control we will use the Pontriagin Maximum principle [38]. First we will determine the Hamiltonian

$$\begin{aligned} \mathcal{H}(\mathbf{x}, \psi, \mathbf{v}) = & -\psi_1 a_1 x_1 - \psi_1 a_2 x_2 x_3 + \psi_1 v_1 \\ & -\psi_2 b_1 x_2 + \psi_2 v_2 + \psi_3 c_1 x_1 x_2 - \psi_3 c_2 x_3 \end{aligned} \tag{20}$$

Maximum principle states, that the optimal control maximizes the Hamiltonian, that is

$$\mathcal{H}(\mathbf{x}^*, \psi^*, \mathbf{v}^*) \geq \mathcal{H}(\mathbf{x}^*, \psi^*, \mathbf{u}) \tag{21}$$

for any vector $\mathbf{u}$ in the admisible set of controls, with $\mathbf{x}^*$, ψ^* describe the appropriate optimal trajectory and optimal adjoint variable. For our system optimality condition takes form

$$\psi_1^* v_1^* + \psi_2^* v_2^* \geq \psi_1^* u_1 + \psi_2^* u_2 \tag{22}$$

And adjoint variable ψ is given by the following differential equation

$$\begin{aligned} \dot{\psi}_1 = & -a_1\psi_1 + c_1 x_2 \psi_3 \\ \dot{\psi}_2 = & -a_2 x_3 \psi_1 - b_1 \psi_2 + c_1 x_1 \psi_3 \\ \dot{\psi}_3 = & -a_2 x_2 \psi_1 - c_2 \psi_3 \end{aligned} \tag{23}$$

with terminal condition

$$\psi(T) = -\nabla q(\mathbf{x}(T)) \tag{24}$$

From the optimality condition (22) we have that the optimal controls are piecewise constant taking values on the boundary of the admissible set. This knowledge allows finding the optimal control by optimising the switching times of control values between maximal and minimal. Numerical solution of this problem using the performance index derivatives obtained using the adjoint equations (23) can be found in [3]. General method of constructing optimal control including the evolution of control signal structure based on switching optimisation can be found in [47].

3. Sampling period optimization in computer control system. In the paper [34] a digital control system for an experimental heat control plant is considered. The control plant is shown in Figure 6.

The structure of the digital control system is shown in Figure 7. In this figure $y^+(k) = y(kh), h > 0, k = 0, 1, 2, ...$ and $u(t) = u^+(k)$ for $t \in [kh, (k+1)h)$, $h > 0$ denotes the sample time of D/A and A/D converters working synchronously.

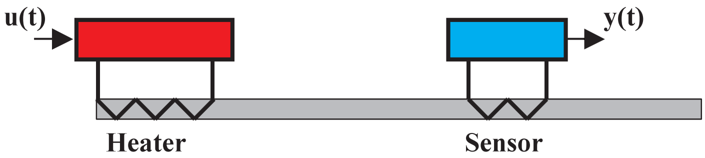

Figure 6: Heat control of a thin copper rod.

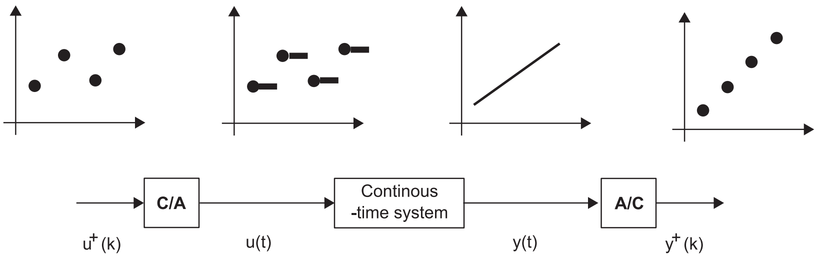

Figure 7: Discrete-continuous system.

Then the dynamic feedback is designed ([25, 26]) in the form $u^+ = F(y^+, r)$, where r is a set point. During tests of this control system it was observed that the settling time T_c (after this time the difference between the set point r and the process value $y(t)$ is stably smaller than 5%) is a function of the sample time $h > 0$, $t_k = kh$, $k = 0, 1, 2, ...$, and this function has a minimum. This means that there exists a value of the sample time minimizing the settling time T_c, which is one of fundamental direct control cost functions, applied in the industrial practice.

In the paper [34] an analytical formula for the settling time was obtained in the form

$$J(h) = \frac{\mathrm{e}^{-R_a h}}{c_0} + \frac{R_a \mathrm{e}^{-R_a h}}{b_0(1 - \mathrm{e}^{-R_a h})} + \alpha h + S \tag{25}$$

Function $J(h)$ is a good approximation (from experiments for $h \in (650[\mathrm{ms}], 1000[\mathrm{ms}])$) of the settling time T_c as a function of sample time h in the considered laboratory control system. Parameters of function $J(h)$ were obtained by model identification ([36, 34]).

The relation (25) was proposed after the analysis of the simple model of the considered control plant. It can be helpful to explain the phenomenon of the existence of optimal sample time h_{opt} observed during experiments with the use of real soft PLC (Programmable Logic Controller) control system.

4. LQ games. Linear-quadratic (LQ) game is a mathematical concept introduced by Starr and Ho in the paper [19]. Its applications are very wide and miscellaneous – we're dealing with such problems in politics (e.g. optimal fiscal policies, see [48]), economy, ecology, engineering, control theory, and many others. The biggest advance of this theory is a possibility of obtaining trajectories for nonzero-sum cases, i.e. situations, when goals of players are not completely opposite.

4.1. *Basics of LQ game theory.* In the 2-person linear-quadratic game theory, linear time invariant systems (26) are taken into account

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_1\mathbf{u}_1(t) + \mathbf{B}_2\mathbf{u}_2(t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{26}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}_i(t) \in \mathbb{R}^r$, and $\mathbf{A}$, $\mathbf{B}_i$ – constant matrices.

Additionally, players optimize quadratic payoff functions in form (27).

$$J = \int_0^T [\mathbf{x}^{\mathcal{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathcal{T}}(t)\mathbf{R}\mathbf{u}(t)]\, dt + \mathbf{x}^{\mathcal{T}}(T)\mathbf{Q}_T\mathbf{x}(T) \tag{27}$$

where $\mathbf{x}(t)$ is a state vector, and $\mathbf{u}_1$, $\mathbf{u}_2$ are controls of players. When we assume the horizon $T \to \infty$, thus (27) will take a form

$$J = \int_0^\infty [\mathbf{x}^{\mathcal{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathcal{T}}(t)\mathbf{R}\mathbf{u}(t)]\, dt \tag{28}$$

It should be noticed that, in general case, each player has his own performance index (27) or (28) being minimized. So, in 2-person game with infinite time horizon, we have two indexes

$$J_1 = \int_0^\infty [\mathbf{x}^{\mathcal{T}}(t)\mathbf{Q}_1\mathbf{x}(t) + \mathbf{u}_1^{\mathcal{T}}(t)\mathbf{R}_{11}\mathbf{u}_1(t)]\, dt \tag{29}$$

and

$$J_2 = \int_0^\infty [\mathbf{x}^{\mathcal{T}}(t)\mathbf{Q}_2\mathbf{x}(t) + \mathbf{u}_2^{\mathcal{T}}(t)\mathbf{R}_{22}\mathbf{u}_2(t)]\, dt \tag{30}$$

We also assume that $\mathbf{Q}_i = \mathbf{Q}_i^{\mathcal{T}}$, $\mathbf{R}_{ii} = \mathbf{R}_{ii}^{\mathcal{T}}$, $\mathbf{R}_{ii} > 0$, $i = 1, 2$.

Now we will introduce a concept of a *Nash equilibrium*. We say that a pair of strategies $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is in the Nash equilibrium, when they satisfies simultaneously

$$J_1(\mathbf{u}_1^*, \mathbf{u}_2^*) \le J_1(\mathbf{u}_1, \mathbf{u}_2^*) \text{ and } J_2(\mathbf{u}_1^*, \mathbf{u}_2^*) \le J_1(\mathbf{u}_1^*, \mathbf{u}_2)$$

for every admissible $\mathbf{u}_1$, $\mathbf{u}_2$. In other words, each strategy in the Nash equilibrium is a best response to all other strategies in that equilibrium.

Existence and form of the solutions (i.e. trajectories in the Nash equilibrium) are strictly connected with solutions of a specific type of nonsymmetric, coupled Riccati equations. For finite horizon cases the equations are differential, when for infinite horizon cases they turn to algebraic ones. Solutions of this equations are necessary for obtaining trajectories of the system, so the main question is how to solve them.

There are several algorithms for solving coupled Riccati equations related with LQ games with both finite and infinite horizon. One can distinguish direct ([14]) and iterative algorithms. Direct methods allow to obtain solutions using properties of game matrices, as eigenvalues and eigenvectors; their biggest disadvantage is a complexity and lack of possibility of obtaining solutions in closed-loop form. The iterative ones are mostly based on Newton's method (see e.g. [11]), and their applications are limited only to positive systems.

Below we present two examples of LQ games and its solutions

4.2. *Electricity market modelling.* A simple model of the electricity market will be based on the following assumptions:

1. Consumer can change the producer of energy, but it requires some effort from both sides. As a result, electricity market seems to be characterized with relatively big inertia
2. The price of energy is partly regulated by Urząd Regulacji Energetyki (URE). In particular, price can not exceed some fixed value, further called *maximal price*
3. Business model for energy producers is based on the long-term strategies, and, besides of profits, must take into account the trust and opinion of the consumer

Basing on above assumptions, we can create linear model as follows:

- x_i (state) represents the level of aversion of a client to the particular producer. Greater level of aversion causes with lower amount of energy purchased by clients
- u_i (control) - the level of discount. As mentioned in assumptions, the price of energy has its upper bound, defined by URE. Nevertheless, producers can sell their energy cheaper. The value of u_i defines how much the price is lower in compare to maximal price, i.e. $u_i = U - C_i$, where U-maximal price, C_i- the price of energy for producer i
- J_i - performance index for i-th producer. The goal of producer is some kind of compromise between maximization of profit and gain of clients' trust.

We will try to get linear differential equation of (26), representing the dynamic of clients' attitude to particular producer. Such model takes into account the inertia of customers' trust, which, as mentioned, appears on the real markets. It can be expected, than elements of matrix $\mathbf{A}$ will be negative on its diagonal and also besides it. The justification of signs depends on the following observations of customers' behavior:

1. Elements on the diagonal represent the influence of current opinion of

the producer on the future opinion. This elements should be negative, because the natural tendency of consumer is to assure them in their opinions (phenomenon of so called *after decision discord*, see e.g. [7]). The choice of a particular producer makes the consumer "convincing themselves" even without any specific actions from the side of the producer.

2. Elements which are not on the diagonal represent the influence of other produrers' opinion on opinion of particular producer. When the opinion is bad in comparing to others, it will cause with further making it worse.

System trajectories for open-loop information model can be obtained using Engwerda's algorithm (see [15]). Using parameters

$$\mathbf{A} = \begin{bmatrix} -1 & -1.3 \\ -1.1 & -1 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}, \qquad \mathbf{B}_2 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}$$

and weight matrices

$$\mathbf{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

we will obtain trajectories presented on the picture (8). It can be noticed that, as expected, optimal strategy for Player 2 (the "greedy" one) is to offer a low discount – it will make his profit greater. Player 1 offers much bigger discount, which results with much better opinion after a short while (to make it more visible, in the example the initial aversion to both producers is identical).

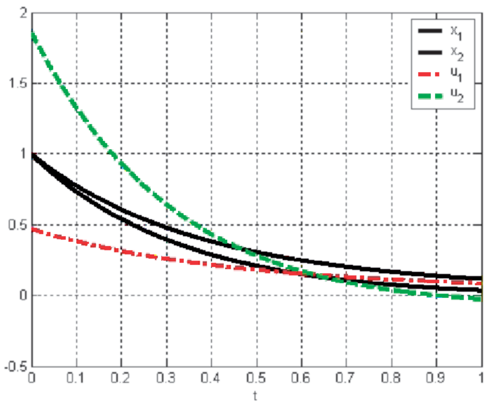

Figure 8: Trajectories in Nash equilibrium

4.3. *Long RC transmission line.* Let's consider a homogeneous rc transmission line, i.e. one where the parameters per the unit length (resistance

r and capacity c are constant and independent of the coordinate $z \in [0, l]$, where l – the length of a line). An infinitesimal part of the long line is described by equation (see e.g. [20], [30]).

$$rc\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2} \tag{31}$$

with boundary conditions

$$x(t,0) = u_1(t)$$
$$x(t,l) = u_2(t)$$
$$x(0,z) = \phi(z)$$

where $t \geq 0$, $0 \leq z \leq l$.

In general, equation (31) is hard to solve. Nevertheless, the system can be approximated by the RC ladder network depicted on figure (9), where $R = rl$, $C = cl$ (see [8], p. 314).

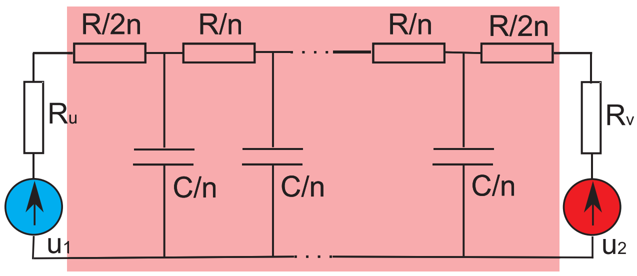

Figure 9: RC ladder scheme

A ladder network depicted on figure (9) is a linear system, that can be described by state equation with two independent inputs.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_1 u_1(t) + \mathbf{B}_2 u_2(t) \tag{32}$$

where $\mathbf{x}(t)$ - state vector, $u_1(t)$, $u_2(t)$ – independent inputs.

Matrix $\mathbf{A}$ is a tridiagonal Jacobi matrix taking form (see [30])

$$\mathbf{A} = [a_{i,j}], \qquad a_{i,j} = 0 \qquad |i-j| > 1$$
$$a_{i,i} = -2\frac{n^2}{RC} \qquad i = 2, 3, \ldots, n-1$$
$$a_{1,1} = -(1 + \frac{2R}{2nR_u + R})\frac{n^2}{RC}$$
$$a_{n,n} = -(1 + \frac{2R}{2nR_v + R})\frac{n^2}{RC}$$
$$a_{i,i-1} = \frac{n^2}{RC} \qquad i = 2, 3, \ldots, n$$

$$a_{i,i+1} = \frac{n^2}{RC} \qquad i = 1, 2, \ldots, n-1$$

and $\mathbf{B}_1$, $\mathbf{B}_2$ take form

$$\mathbf{B}_1 = \frac{2n^2}{C(2nR_u + R)}\mathbf{e}_1 \qquad \mathbf{e}_1 = [1\ 0\ 0\ \ldots\ 0] \tag{33}$$

$$\mathbf{B}_2 = \frac{2n^2}{C(2nR_v + R)}\mathbf{e}_2 \qquad \mathbf{e}_2 = [1\ 0\ 0\ \ldots\ 1] \tag{34}$$

For fixed n the tridiagonal Jacobi matrix $\mathbf{A}$ has only single eigenvalues λ_i. The matrix $\mathbf{A}$ is diagonalizable. The Jordan canonical form of $\mathbf{A}$ is $J = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. From Gershgorin's criterion and the fact that $\det \mathbf{A} \neq 0$, we have $\lambda_i \in [-m, 0)$, where $m = \max_i(|a_{i,i-1} + a_{i,i+1}|)$. Thus the system (32) is asymptotically stable (see [30]).

Because of positiveness of the system, we can use iterative Newton's algorithm to obtain system trajectories (see [16]). For further analysis we will use the following system parameters

$$R_{u1} = R_{u2} = 1$$
$$R = 2$$
$$C = 2$$

and weight matrices

$$\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{I}_{n\times n}$$

Results for 3-dimensional infinite-horizon case are depicted on figure (10). As we can see, all trajectories are asymptotically stable. It can also be noticed that both players are trying to stabilize the system, but input of the first player is much greater than the second one. This is caused with the fact that control cost is greater for the second player.

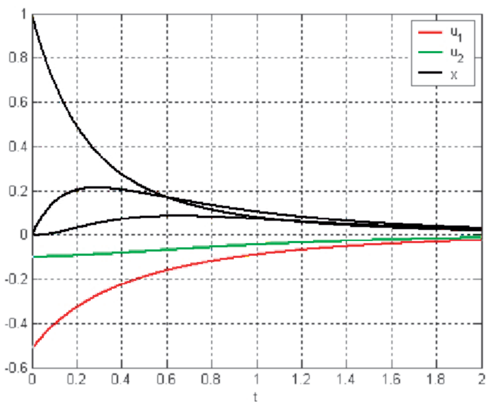

Figure 10: Trajectories in Nash equilibrium

5. Shape optimization. The problem of shape optimization consists of finding a shape (in two or three dimensions), which is optimal in a certain sense and satisfies certain requirements. The problem typically involves the solution of a system of nonlinear partial differential equations, which depend on parameters that define a geometrical domain. The solution is usually obtained numerically, by using iterative methods, for example, by the finite element methods [45]. An interesting approach has been developed by [46, 42, 43] for some kind of the shape optimization problems. They have considered a single span beam with rectangular cross-section working under self-weight (Fig. 11).

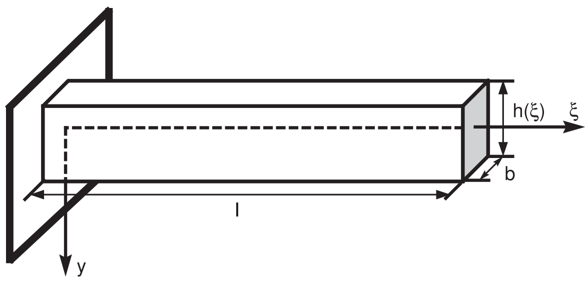

Figure 11: Single span beam under self-weight

The statics of the beam can be described by the following equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(EI(\xi)\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}\right) = -u(\xi), \tag{35}$$

where $\xi \in [0, l]$, $u(\xi) = \gamma b h(\xi)$, $y(\xi)$ represents vertical displacement of the beam along the interval $[0, l]$, l stands for the length of the beam, b is the width of the beam's cross-section, h is the height of the cross-section, E is the Young's module, $I(\xi)$ is the moment of inertia of the cross-section, γ is the volume mass density of the beam material. Side conditions concerning strength and geometry are imposed on the dimensions of the cross-section, so that

$$H_1 \leq u(\xi) \leq H_2, \quad \xi \in [0, l]. \tag{36}$$

The deflection at the end point of the beam is the optimality criterion

$$J(u) = y(l). \tag{37}$$

The problem is to determine such u, which minimizes the functional (37) and satisfies the equation (35) with the boundary conditions (36). The Pontryagin maximum principle [39, 6] can be used for solving the formulated task of optimization. Using this method, the numerical algorithm has been designed and implemented [42, 43]. The algorithm uses also an iterative me-

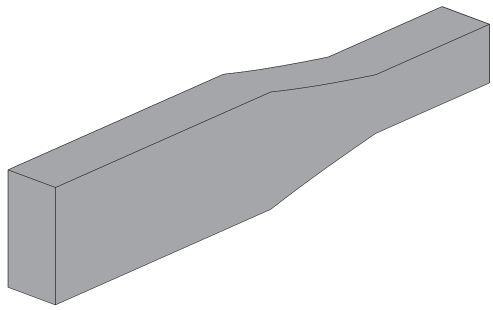

Figure 12: Optimum shape of the beam

thod, that is, it starts with an initial guess of a shape, and then gradually evolves it, until it falls into the optimum shape (Fig. 12).

6. Feedback stabilization of distributed parameter gyroscopic systems. Gyroscopic systems represent a large class of physical systems and also form an integral component in many engineering applications [18]. Power generation systems using vibration of moving bodies contains gyroscopic effect. Mechanical systems undergoing rotation about a fixed axis are generally subject to gyroscopic forces, which are associated with coriolis accelerations or mass transport. For example, the motion of a taut string, rotating about its ξ-axis with constant velocity ω (Fig. 13), is described by the system of partial differential equations:

$$\frac{\partial^2 x_1(t,\xi)}{\partial t^2} - 2\omega\frac{\partial x_2(t,\xi)}{\partial t} - \omega^2 x_1(t,\xi) - \frac{\partial^2 x_1(t,\xi)}{\partial \xi^2} = b(\xi)u(t), \tag{38}$$

$$\frac{\partial^2 x_2(t,\xi)}{\partial t^2} + 2\omega\frac{\partial x_1(t,\xi)}{\partial t} - \omega^2 x_2(t,\xi) - \frac{\partial^2 x_2(t,\xi)}{\partial \xi^2} = 0, \tag{39}$$

with the boundary conditions

$$x_1(t,0) = x_1(t,1) = x_2(t,0) = x_2(t,1) = 0, \tag{40}$$

where $t > 0$, $\xi \in (0,1)$. The function $b : [0,1] \to \mathbb{R}$ represents the control force applied to the system.

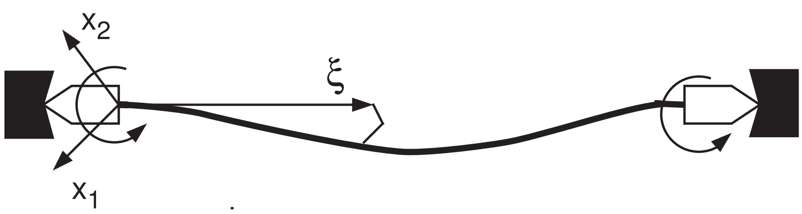

Figure 13: Small oscillations of a taut rotating string

The system (38), (39) has an infinite number of poles on the imaginary axis and is not asymptotically stable. In the paper [40], two types of control law have been presented:

$$u(t) = -k_1 y(t) - k_2 \dot{y}(t), \quad k_1 > 0,\, k_2 > 0, \tag{41}$$

and

$$u(t) = -k\left(w(t) + y(t)\right), \quad k > 0, \tag{42}$$

where

$$\dot{w}(t) + \alpha w(t) = \beta u(t), \quad \alpha > 0,\, \beta > 0, \tag{43}$$

and

$$y(t) = \int_0^1 b(\tau) x_1(t,\tau)\, \mathrm{d}\tau. \tag{44}$$

The designed controllers are finite dimensional and capable to asymptotically stabilize the system in a wide range of their parameters. Reduced-order design of the controllers and their robustness can be considered as the main advantages of the approach.

7. Stabilisation of LC ladder network with time delay controller.

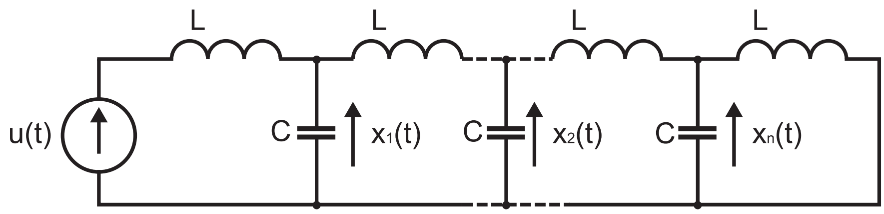

Figure 14: LC ladder network

Mathematical model of LC ladder with $y(t) = x_n(t)$ depicted at figure 14 is given by a following second order equation of dimension n

$$\ddot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) = \mathbf{B}u(t) \tag{45}$$
$$y(t) = \mathbf{C}\mathbf{x}(t) \tag{46}$$

where $\mathbf{x}(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R},\ y(t) \in \mathbb{R}$ and

$$\mathbf{A} = \frac{1}{LC}\begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}_{n\times n} \tag{47}$$

$$\mathbf{B} = \frac{1}{LC}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \qquad \mathbf{C} = \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \end{bmatrix}_{1\times n}$$

In this paper we will consider a feedback in the form of proportional, time delayed controller

$$u(t) = Ky(t-h) \tag{48}$$

where $K \in \mathbb{R}$ is the gain and $h > 0$ is the time delay. Usually, in control application focus is on elimination of the influence of delay (which is usually negative), what leads to difficult control problems. On the other hand, introducing or increasing a delay to the system is very simple - it can be implemented with appropriate buffers. That is why such controller can be easily applied. Equation of closed loop system is given by

$$\ddot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) - \mathbf{B}K\mathbf{C}\mathbf{x}(t-h) = 0 \tag{49}$$

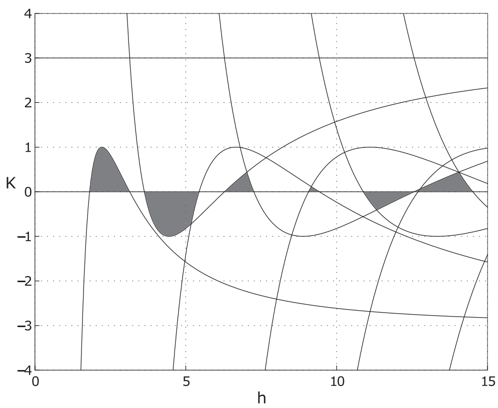

Figure 15: Stability regions for $n = 2$

Using the method of D-partition and computing the characteristic quasi-polynomial one can find pairs of K and h which will stabilise this system. Example of such "region of stability" for ladder with $n = 2$ is presented in the figure 15.

8. Inhomogeneous electric ladder networks with nonlinear elements. Electric ladder networks may be described as networks formed by numerous repetitions of an elementary cell. The elementary cell may consist of resistors, inductance coils, and capacitors connected in series or in parallel. If all the elementary cells are identical, the ladder network is said to be homogeneous; if the elementary cells are not identical, the ladder network is called inhomogeneous. Electric ladder networks may be employed to model both electrical and nonelectrical systems with distributed parameters. They may be used to calculate the voltage distribution in insulator string and in the windings of electric machines and transformers. They may also be employed to compute the pressure distribution in mechanical and thermal systems with distributed parameters. Ladder networks composed of reactive elements, such as inductance coils and capacitors, are used as artificial delay lines, in which the output signal lags behind the input signal; in such delay lines, the delay time is determined by the network parameters. Ladder networks are also deployed as electric filters.

Stabilization of homogeneous electric ladder networks is widely discussed in [24, 29, 31]. Feedback stabilization problem for an inhomogeneous RC ladder network with nonlinear elements has been considered recently in [41]. Such circuit (Fig. 16) consists of a set of resistors R_i, capacitors C_i and a voltage source $u(t)$. The resistors and capacitors have in general nonlinear characteristics.

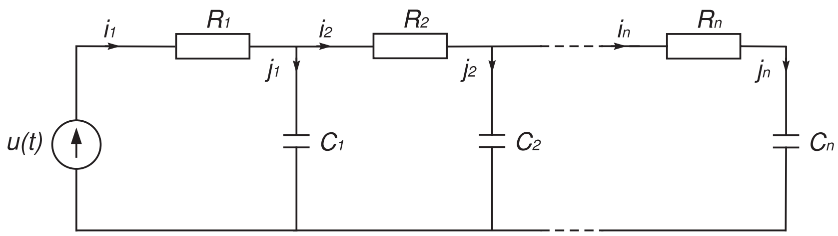

Figure 16: Schematic diagram of an inhomogeneous RC ladder network

The circuit's dynamic behavior can be governed by the following equation:

$$\mathbf{R}(\dot{\mathbf{p}})\dot{\mathbf{p}}(t) + \mathbf{C}(\mathbf{p})\mathbf{p}(t) = \mathbf{B}u(t), \tag{50}$$

where

$$\mathbf{R}(\dot{\mathbf{p}}) = \operatorname{diag}(R_1(\dot{\mathbf{p}}), R_2(\dot{\mathbf{p}}), \ldots, R_n(\dot{\mathbf{p}})), \tag{51}$$

$$\mathbf{C}(\mathbf{p}) = \begin{bmatrix} d_1 & e_2 & 0 & \dots & 0 & 0 \\ f_2 & d_2 & e_3 & \dots & 0 & 0 \\ 0 & f_3 & d_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} & e_n \\ 0 & 0 & 0 & \dots & f_n & d_n \end{bmatrix}, \tag{52}$$

where

$$d_1 = \frac{1}{C_1(\mathbf{p})}, \tag{53}$$

$$d_k = \frac{1}{C_{k-1}(\mathbf{p})} + \frac{1}{C_k(\mathbf{p})}, \tag{54}$$

$$e_k = f_k = -\frac{1}{C_{k-1}(\mathbf{p})}, \tag{55}$$

for $k = 2, 3, \dots, n$,

$$\mathbf{B} = [\,1 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0\,]^{\mathrm{T}}\,, \tag{56}$$

$$\mathbf{p}(t) = [\,p_1(t) \quad p_2(t) \quad \dots \quad p_n(t)\,]^{\mathrm{T}}\,, \quad i_k(t) = \dot{p}_k(t). \tag{57}$$

It can be shown [41], that an uncontrolled system is already asymptotically stable and the dynamic feedback

$$u(t) = -\frac{w(t) + p_1(t)}{K_0 + \gamma\,(w(t) + p_1(t))^2}, \quad K_0 > 0,\ \gamma \geq 0, \tag{58}$$

$$\dot{w}(t) + \alpha w(t) = \beta u(t), \quad w(0) = w^0,\ w^0 \in \mathbb{R},\ w(t) \in \mathbb{R}, \tag{59}$$

can improve the dynamic stability performance of the corresponding closed-loop system.

9. Summary. In this paper we presented a selection of advanced applications of mathematics in control and game theory setting. Among the others we have discussed practical control problems with an example of DC motor, decision support application for electric market modelling, stabilisation schemes for finite and infinite dimensional systems. All these examples are interesting areas of research and at the same time present only a fraction of possibilities of control and game theory which can be later used in real life solutions.

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Zastosowania matematyki w wybranych układach sterowania i procesach decyzyjnych

Streszczenie. Na styku teorii i praktyki pojawia się coraz częściej nowa dyscyplina naukowa nazywana matematyką przemysłową lub technomatematyką. Nie jest to pomysł nowy. Historia nauki od dawna obserwuje usytuowanie rozważań matematycznych pomiędzy Światem abstrakcyjnych idei a światem materialnym. Ten fakt dobrze oddaje znana myśl Hugo Steinhausa: „Między duchem a materią pośredniczy matematyka” [44]. Obszarem matematyki przemysłowej jest modelowanie różnego typu obiektów rzeczywistych i następnie poszukiwanie odpowiednich metod numerycznych do rozwiązywania zbudowanych wcześniej modeli matematycznych. W konsekwencji otrzymujemy algorytmy wspomagające podejmowanie decyzji w konkretnych procesach przemysłowych. Pomiędzy dobrą teorią i praktyką występuję pewnego rodzaju sprzężenie zwrotne. Teoria pozwala skutecznie oddziaływać na świat materialny. Z kolei rozwiązania techniczne generują nowe problemy matematyczne. Burzliwy rozwój technik komputerowych umożliwił w ostatnich latach dokładniejszą analizę i syntezę układów sterowania złożonymi procesami oraz wspomaga podejmowanie decyzji w różnych obszarach stosowanych praktycznie. Odkrywanie matematycznej struktury świata pobudza przedstawicieli nauk technicznych do działania zmierzającego do celowego oddziaływania na obiekty rzeczywiste. Weryfikacja praktyczna pomysłów inżynierów w wielu przypadkach jest skuteczna i przynosi wymierne efekty. W sterowaniu układów dynamicznych z powodzeniem stosuje się często (nie jest to jedyny sposób postępowania) następujący algorytm działania (zob. np. [27, 28, 32, 33]):

1. Tworzy się model matematyczny, zwykle w postaci odpowiedniego równania różniczkowego.
2. Dokonujemy linearyzacji.
3. Projektujemy układ sterowania, np. poprzez odpowiednie sprzężenie zwrotne. Zwykle formułując odpowiedni problem LQ (problem liniowo kwadratowy).
4. Dokonujemy weryfikacji naszych działań na obiekcie rzeczywistym.

Praktycznie na każdym etapie można przeprowadzać identyfikację parametrów odpowiedniego modelu. Zaprojektowany układ sterowania powinien posiadać odpowiednie własności. Wymagana jest asymptotyczna stabilność (wykładnicza) z odpowiednim obszarem przyciągania (Zasada LaSalle'a [22], s. 64). Wykorzystuje się różne pojęcia stabilność, zwykle w sensie Lapunowa, ([22] s. 34, 61) również praktyczną stabilność ([22] s. 127). Dobrze jest, by zaprojektowany układ zachował typowe własności spotykane w teorii sterowania (np. [26], s. 69, 76, 86, 90), takie jak sterowalność i obserwowalność (stabilizowalność i wykrywalność). Przy sterowaniu komputerowym układ ciągły w czasie współpracuje z urządzeniami pracującymi dyskretnie w czasie (np. z komputerem, sterownikami cyfrowymi, itp.) poprzez odpowiednie przetworniki sygnałów A/C i C/A (przetwornik analogowo-cyfrowy i cyfrowo-analogowy). Przy sterowaniu komputerowym jakość pracy układu zależy od sposobu pracy przetworników A/C i C/A (praca synchroniczna lub praca nie synchroniczna, od wielkości kroku dyskretyzacji czasu, od rozłożenia w przestrzeni poszczególnych urządzeń, itp.). Przy wyznaczaniu parametrów sprzężenia zwrotnego (również dynamicznego) wykorzystuje się odpowiednie równania Lapunowa i Riccatiego (np. [1], lub zob. np. [26,27], [21]), co ma związek z odpowiednimi problemami LQ (np. [17]). Podstawowa filozofia projektowania układów sterowania z wykorzystaniem metody linearyzacji jest zawarta w twierdzeniu Grobmana-Hartmana (np. [35]). Okazuje się, że jeżeli macierz stanu układu liniowego przybliżenia nie posiada wartości własnych na osi urojonych (oczywiście mówimy teraz o przypadku skończenie wymiarowym), to liniowe przybliżenie i układ nieliniowy w pewnym otoczeniu zera zachowuje się „podobnie”

(charakter zachowania trajektorii stanu jest taki sam, dokładniej pomiędzy trajektoriami układów istnieje w pewnym otoczeniu zera homeomorfizm, czyli odpowiednie odwzorowanie wzajemnie jednoznaczne). Analizując asymptotyczną stabilność obszar przyciągania do zera można próbować wyznaczać wykorzystując Zasadę LaSalle'a. Praca ma charakter przeglądowy i zawiera wybrane przykłady wcześniej rozważane przez autorów opracowania. Między innymi krótko omówiono: problemy sterowania silnikiem prądu stałego (rozdział 2), problemy sterowania komputerowego (rozdział 3), zagadnienia wspomagania decyzji przy modelowaniu rynku energii elektrycznej z wykorzystaniem teorii gier (rozdział 4), pewien problem optymalizacji kształtu (rozdział 5) z wykorzystaniem Zasady Maksimum Pontryagina ([39]), problemy stabilizacji systemów skończenie i nieskończenie wymiarowych (rozdział 6, 7 i 8). W przedstawionych przykładach wykorzystano różnorodny aparat matematyczny i w konsekwencji różne metody rozwiązania.
Słowa kluczowe: matematyka przemysłowa, sterowanie, asymptotyczna stabilność, sprzężenie zwrotne, teoria gier, silnik prądu stałego.

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